Instructions:

Write the answers and show all your work in the blue books. There are 4 problems. Make sure you do all 4. No books, notes, or collaboration with others.

Problem 1. (20 points) A function \( f \) is twice differentiable and satisfies the following:

(i) \( f'' > 0 \) on \( (-\infty, -1) \); \( f'' < 0 \) on \((-1,3)\) and \((3, \infty)\).

(ii) \( x = 2 \) and \( 3 \) are critical points.

(iii) \( f' > 0 \) on \((-\infty, -1)\), \((-1,2)\), and \((3, \infty)\); \( f' < 0 \) on \((2,3)\).

(iv) \( \lim_{x \to -1^+} f(x) = -\infty \) and \( \lim_{x \to -1^-} f(x) = +\infty \).

(v) \( f(0) = 0, f(2) = 1, f(3) = 0 \). Also, \( \lim_{x \to \pm \infty} f(x) = 1 \).

From this information, answer the following questions. Show your reasoning in each case:

(a) Determine all local maxima and minima.

(b) Determine all asymptotes, both horizontal and vertical.

(c) Sketch the graph of \( f \).

Problem 2. (8 points) Find the maximum and minimum values of

\[
f(\theta) = \cos(\theta) + \sqrt{3}\sin(\theta)\]

on the interval \([0, \pi]\).

Problem 3. (6 points) A jubjub tree produces one fruit each growing season for each foot of height. Fruit sell for \$10 apiece. However, due to the shadow it casts, a jubjub tree \( x \) feet tall uses up \( 5x^2 \) square feet of orchard space. Assuming rental of orchard space for the growing season is 2 cents per square foot, how tall should you grow your jubjub trees to maximize profits?

Problem 4. (4 points) Assume the function \( f \) is differentiable and \( f'(x) \geq -1 \) for every \( x \). Prove that the inequality

\[
f(x) + x \leq f(y) + y
\]

holds whenever \( x < y \). (Hint: rearrange the inequality and use the Mean Value Theorem.)