Why Can't I Divide by Zero ?

Many people get rebellious when they are told they cannot divide by zero. "Why can't I divide by zero?", they ask. Nobody likes being told they can't do something. But the truth of the matter is, they can! Anyone can define things like $1/0$ to be any number they like. It's a free country.

I'm reminded of the old story about the Indiana legislature and its resolution to make π equal to 3. Generations of schoolchildren had suffered through arithmetic complications brought on by the inconvenient irrational value of π. What better way to ensure their place in history, the lawmakers reasoned, than to remove this burden from subsequent generations? Of course they were entirely within their rights to introduce a funny Greek symbol to stand for the number 3 and to require its use. What the lawmakers failed to understand is that legislating the value of π does not change the value of the ratio of the circumference of a circle to its diameter. That ratio is not subject to change. It is what it is, for better or worse.

The same goes for dividing by zero. You can do it, if you wish, but you will give something up in the bargain, just as the lawmakers in Indiana would have given up the option of using the symbol π to stand for the ratio of circumference to diameter. (Their resolution was not successful.) In the case of division by zero, what must be given up are the familiar laws of arithmetic. So I'm not going to try to convince you in what follows that you *cannot* divide by zero. I shall merely try to convince you that you *should not* divide by zero.

All the familiar rules of arithmetic can be summarized in about 11 rules called the *Field Axioms.* An axiom is an assumption that everyone agrees to believe without proof, and from which all other mathematical facts are derived. The field axioms include the familiar commutative and associative laws of addition and multiplication, the distributive law, as well as the seemingly innocuous requirement that

$$1 \neq 0. \qquad (*)$$

I should point out that the 1 and 0 in this statement do not necessarily refer to the familiar numbers from arithmetic. It is one of the remarkable discoveries of modern mathematics that there are alternative "number systems" which equally well satisfy the field axioms. The simplest example is a number system with only two "numbers" in it, denoted 1 and 0. All the usual rules of arithmetic hold except that $1 + 1 = 0$.

One of the cardinal principles of mathematical reasoning is that facts derived from axioms cannot contradict any of the axioms themselves. If this ever happened, the set of axioms would have to be modified or abandoned. Unfortunately, this is just what happens if one attempts to introduce additional axioms that define values for expressions like $1/0$.

For example, suppose we attempted to define $1/0 = 1$. Then multiplying both sides by zero would give

$$0 \times \frac{1}{0} = 1 \times 0.$$

Cancelling zero on the left and using $1 \times 0 = 0$ on the right, we obtain 1 = 0. Oops! We just violated axiom (*).

What about 0/0 ? This expression arises in connection with the study of limits in calculus, and such limits would be much easier to handle if we had a value for 0/0. Let's try defining $\frac{0}{0} = 1$. If we then multiply both sides by -1 we would have

$$-1 \times (\frac{0}{0}) = -1 \times 1 = -1,$$

or

$$\frac{-1 \times 0}{0} = -1.$$

Since $-1 \times 0 = 0$, we would conclude that $\frac{0}{0} = -1$. But we *defined* this fraction to have the value 1, so we must (reluctantly) conclude that $-1 = 1$. Trouble is looming on the horizon. Indeed, if we add 1 to both sides and then divide by 2 we're right back to $1 = 0$.

A similar difficulty arises if $\frac{0}{0}$ is defined to be any nonzero number. As a last resort, let's try defining

$$\frac{0}{0} = 0.$$

This doesn't work either, but the trouble lies a bit deeper this time. First, we must understand that division isn't really an independent operation at all. You won't find division mentioned anywhere in the 11 field axioms that summarize all the important properties of arithmetic. Indeed, the expression $\frac{a}{b}$ is merely shorthand for the expression $a \times b^{-1}$, where the field axioms guarantee the existence of a *multiplicative inverse* b^{-1} for any $b \neq 0$. It is unique for a given b and satisfies $b \times b^{-1} = 1$.

The field axioms don't directly *forbid* 0 to have a multiplicative inverse, but it turns out they do so indirectly. When we say $\frac{0}{0} = 0$, we're really saying two things: first, that 0 does have a multiplicative inverse after all; and secondly, that it satisfies $0 \times 0^{-1} = 0$. But also $0 \times 0^{-1} = 1$, since that's what it means to be a multiplicative inverse. Thus, 1 = 0. Another way to reach the same conclusion is using the usual rules for adding fractions:

$$\frac{1}{1} + \frac{0}{0} = \frac{0 \times 1 + 1 \times 0}{0 \times 1} = \frac{0}{0} = 0.$$

On the other hand,

$$\frac{1}{1} + \frac{0}{0} = 1 + 0 = 1.$$

Once again, we have $0 = 1$. (Thanks to Andy Vogel and Dan Zacharia for this last example.)

In conclusion I should remark that axiom (*) really has only one purpose: to rule out the trivial example of a field with only 1 element, 0. Why not then simply allow this as an example of a field? Little would be gained by doing this, and much would be lost. Virtually every theorem proved about fields would have have to include a special proviso to handle the exceptional example.