The Associative Law and Riemann's Theorem

Terry R. McConnell
Syracuse University

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Abstract

We comment on divergent series whose terms can be grouped so as to produce an arbitrarily specified sum.

1 Results

According to Riemann's theorem on rearrangements, a conditionally convergent series can be rearranged to have any sum whatever. This shows that the commutative law of addition fails to generalize to infinite sums – only absolutely convergent series can be rearranged at will without affecting the sum.

The associative law, on the other hand, appears to have a satisfactory generalization: the terms of a convergent infinite series can be grouped in an arbitrary way without affecting convergence or the sum. Regrouping the terms only serves to extract a subsequence of the sequence of partial sums.

The author has often made these comments, or ones like them, in his calculus classes, unaware that the comment about the associative law reveals a clear bias in favor of convergent series. Terms of divergent series can often be grouped so as to produce convergent series. Take, for example, the series with terms $(-1)^n, n = 0, 1, 2, \ldots$. Grouping the terms in pairs turns a divergent series into a convergent series.

Indeed, it is not uncommon for a divergent series to behave in a way that is analogous to the series involved in Riemann's theorem, but relative to grouping of terms rather than rearrangement. Let us say that an infinite series is *generative* if its set of partial sums is dense in the set of real numbers. (Such series are necessarily divergent.)

Lemma 1.1. *Let s_n be the n-th partial sum of a series $\sum_{j=1}^{\infty} b_j$ for which we have that $|b_j| \to 0$ as $j \to \infty$, and such that*

$$\limsup_{n\to\infty} s_n = +\infty \text{ and } \liminf_{n\to\infty} s_n = -\infty.$$

Then $\sum_{j=1}^{\infty} b_j$ is generative.

For the proof, given any $\alpha \in \mathbb{R}$, let $T_1 = \inf\{n : s_n > \alpha\}$ and $T_2 = \inf\{n > T_1 : s_n < \alpha\}$. The hypotheses ensure that T_1 and T_2 are both well-defined and finite. Continue inductively to define a sequence of indices in pairs: $T_{2k+1} = \inf\{n > T_{2k} : s_n > \alpha\}$ and $T_{2k+2} = \inf\{n > T_{2k+1} : s_n < \alpha\}$. Then we have $s_{T_n} \to \alpha$ as $n \to \infty$, since $|b_{T_n}| \to 0$ as $n \to \infty$.

Theorem 1.1. *Any conditionally convergent series has a rearrangement that is generative.*

To see this, produce a rearrangement that satisfies the hypotheses of Lemma 1.1 by proceeding as in the usual proof of Riemann's theorem: alternate longer and longer strings of positive terms with longer and longer strings of negative terms, so as to exhaust all the terms. See, for example [3], pp. 76-7.

The result of Lemma 1.1 also holds if we assume only $b_j^+ \to 0$ or $b_j^- \to 0$ as $j \to \infty$. One may proceed as in the proof of Lemma 1.1, using either the odd indexed s_{T_n} or the even indexed s_{T_n}. It is not difficult to produce examples showing that neither condition is necessary. On the other hand, we do have the following result:

Theorem 1.2. *The terms of any generative series* $\sum_{j=1}^{\infty} b_j$ *can be grouped so as to produce a new generative series* $\sum_{j=1}^{\infty} c_j$ *for which* $|c_j| \to 0$ *as* $j \to \infty$.

To see this, fix a generative series $\sum_{j=1}^{\infty} a_j$ for which we do have $|a_j| \to 0$ as $j \to \infty$, using, e.g., Theorem 1.1. Now, it holds that $\sum_{j=k}^{\infty} b_j$ is also generative for any given k, since translates of dense subsets of $\mathbb{R}$ are dense. Using this observation, we may inductively define the c_j as sums of blocks of terms of $\sum_{j=1}^{\infty} b_j$ such that $|c_j - a_j| < 2^{-j}$. The desired result then follows from:

Lemma 1.2. *If* $\sum_{j=1}^{\infty} a_j$ *is generative and* $\sum_{j=1}^{\infty} |a_j - c_j| < \infty$, *then* $\sum_{j=1}^{\infty} c_j$ *is generative.*

For the proof, given $\alpha \in \mathbb{R}$ and $\epsilon > 0$, first find M such that $\sum_{j=M+1}^{\infty} |a_j - c_j| < \frac{\epsilon}{2}$. Since $\sum_{j=M+1}^{\infty} a_j$ is generative, a subsequence of its partial sums converges to $\alpha - \sum_{j=1}^{M} c_j$. In particular, there is N such that $|\sum_{j=M+1}^{N} a_j - \alpha + \sum_{j=1}^{M} c_j| < \frac{\epsilon}{2}$. But then $|\sum_{j=1}^{N} c_j - \alpha| < \epsilon$.

It is also easy to obtain generative series as random series. For example:

Theorem 1.3. *Let* $X_n, n = 1, 2, \ldots$ *be independent symmetric random variables such that*

$$\sum_{n=1}^{\infty} E(X_n^2) = \infty$$

and

$$\sum_{n=1}^{\infty} E|X_n|^p < \infty$$

for some $p > 2$. *Then* $\sum_{n=1}^{\infty} X_n$ *is generative with probability one.*

This follows from the Kolmogorov Three Series Theorem. See, for example, [1], p. 114. (The convergent series hypothesis can be replaced by any condition that ensures $X_n \to 0$, a.s.)

More highly divergent examples of generative random series arise from neighborhood recurrent random walks. Take for the X_n an i.i.d sequence whose distribution is non-lattice and satisfies the Weak Law of Large Numbers. The resulting random series is almost surely generative by the Chung-Fuchs Theorem. See, for example, section 3.2 of [2]. It would be of interest to find necessary and sufficient conditions for a series of independent, but not identically distributed, random variables to be generative with probability one.

References

[1] Y.Chow and H. Teicher, *Probability Theory*, Springer Verlag, New York, 1978.

[2] R. Durrett, *Probability: Theory and Examples*, 3rd Edition, Brooks-Cole, Belmont, CA, 2005.

[3] Walter Rudin , *Principles of Mathematical Analysis*, 3rd Edition, McGraw-Hill, New York, 1976.